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Note

Least-Squares Solution for the Blunt Body Hypersonic Flow Problem

The problem of the detached shock wave has recently received considerable attention [2, 3]. In this note a variational method [4], the discrete nonlinear least squares method is applied to such a problem. The objective of this particular numerical experiment is to provide a first approximation to the solution for the flow of gas in a coaxial plasma accelerator in the M.H.D. approximation. This problem involves a free boundary of the "body" but with a known pressure distribution [5].

The basic idea of a variational approach is the transformation of a boundary value problem into a problem of finding the minimum of certain functionals. This is performed completely numerically for ideal gas hypersonic flow around a sphere, whose solution is known [1]. This problem may be written in the form of three differential equations for unknowns $Y = (u, v, p)$ with independent variables $x = (r, \theta)$:

$$\begin{aligned} f_1(x, Y) &\equiv r(\rho v)_r + (\rho u)_\theta + \rho(2v + u \cot an\,\theta) = 0, \\ f_2(x, Y) &= rp_r/\rho + rvv_r + uv_\theta - u^2 = 0, \\ f_3(x, Y) &= p_\theta/\rho + uu_\theta + rvu_r + uv = 0, \end{aligned} \tag{1}$$

where partial derivatives are denoted by subscripts and where

$$\rho = \frac{\gamma}{\gamma - 1}\frac{p}{h}, \qquad h_{\text{tot}} = h + \frac{u^2 + v^2}{2} = \frac{1}{2}. \tag{2}$$

The quantities u, v, p, ρ and h are two components of velocity, pressure, mass density and enthalpy respectively, normalized to $v_{\max}$, $v_{\max}$, $\rho_0 v_{\max}^2/2$, ρ_0 and $v_{\max}^2$, respectively. The subscript 0 is related to free stream values, velocity $v_{\max}$ is related to total enthalpy h_{tot}. The boundary conditions are

$$v = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad u = 0 \quad \text{for} \quad \theta = 0, \tag{3}$$

$$Y_{\text{shock}} = Y_{\text{shock}}(p_0, h_0, v_0, \sigma), \tag{4}$$

where Y_{shock} denotes the value just behind the shock wave. The angle σ is shown on Fig. 1a. Conditions (5) are called Hugoniot relations [1]. Note that it is not

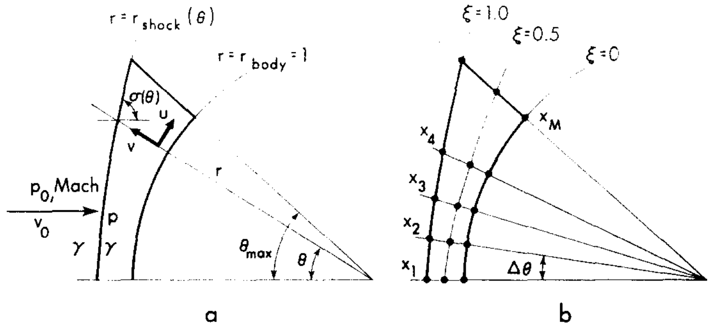

FIG. 1. Diagram of the axisymmetric flow system showing the variables and notation used. In (b), the variable r is replaced by $\xi = (r - r_{\text{body}})/(r_{\text{shock}} - r_{\text{body}})$. The region of interest is represented by the points $x_1, x_2, \ldots, x_M$.

possible in this particular case to write the boundary conditions for $\theta = \theta_{\max}$, which are required if the finite difference approach is to be employed.

A great deal of theoretical work has been done on the problem of finding sufficiently wide classes of functions in order that some functionals actually attain their minimum [4]. This note presents two modifications of [4]:

(i) The class of problems considered will be the one for which the approximate form of the solution, $\tilde{Y}(x, A) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, is known.

For example, let

$$\begin{aligned} u \approx \tilde{y}_1 &= A_1\theta + A_2\theta^3 + A_3\theta^5 + (A_4\theta + A_5\theta^3 + A_6\theta^5)\xi \\ &\quad + (A_7\theta + A_8\theta^3 + A_9\theta^5)\,\xi^2, \\ v \approx \tilde{y}_2 &= A_{10} + A_{11}\theta^2 + A_{12}\theta^4 + (A_{13} + A_{14}\theta^2 + A_{15}\theta^4)\xi \\ &\quad + (A_{16} + A_{17}\theta^2 + A_{18}\theta^4)\xi^2, \\ p \approx \tilde{y}_3 &= A_{19} + A_{20}\theta^2 + A_{21}\theta^4 + (A_{22} + A_{23}\theta^2 + A_{24}\theta^4)\xi \\ &\quad + (A_{25} + A_{26}\theta^2 + A_{27}\theta^4)\xi^2, \\ r_{\text{shock}} &\approx A_1{}^* + A_2{}^*\theta^2 + A_3{}^*\theta^4. \end{aligned} \tag{5}$$

(ii) In this analysis of the detached shock wave problem, the position and shape of one part of the boundary (shock wave) is given implicitly by the Hugoniot relations (4).

The method proposed differs from the method of integral relations [1], Scheme III, in the form of the functions given by (5) and in the method by which A and A^* are determined.

Variational Formulation

Let A^* and the points x be given. The solution of the boundary value problem [Eqs. (1)–(4)] is given by the set A for which the functional given by

$$F(A) \equiv \sum_{\substack{k=1,3\\ i=1,M}} w_k^2 \cdot f_k^2(x_i, \tilde{Y}) \tag{6}$$

is minimal. The determination of weighting factors w_k will be discussed later.

A second functional

$$F^*(A^*) = F(A(A^*)) \tag{7}$$

is introduced for obtaining the solution for A^* under assumption that the correct shock shape and position is obtained when F^* is minimized.

The usual method [4] for minimizing F and F^* involves the solution of equations

$$\partial F/\partial A_l = 0, \quad l = 1, 2, \ldots, N \qquad \text{and} \qquad \partial F^*/\partial A_k^* = 0, \quad k = 1, 2, 3, \tag{8}$$

respectively, where the derivatives are replaced by numerical differentiation. If N is set to 27 in (5), the boundary conditions (3) and (4) may be used to reduce N to 15. Providing a reasonable first guess for A and A^* is available, the solution to Eqs. (3) and (4) may be obtained by the two level minimization of F and F^* using some generalized Newton method (e.g., Newton–Kantorovich method [4]). The convergence of the above method however was found to be slow. An alternate technique was used to accomplish the minimizations:

(i) Minimization of $F(A)$. A more rapid convergence can be obtained if a better approximation to the minimized functional is available. This may be achieved by linearizing the functions $f_k[x_i, \tilde{Y}(x_i, A)]$, $k = 1, 2, 3$; $i = 1, 2, \ldots, M$; rather than linearizing the functions of $\partial F/\partial A_l$; $l = 1, 2, \ldots, N$; in (6). This is performed by using the weighted nonlinear least squares approach to solve the system of $3M$ equations

$$f_k[x_i, \tilde{Y}(x_i, A)] = 0 \tag{9}$$

for $A_1, A_2, \ldots, A_N$. This is accomplished by iterative improvements of a given first guess A^G through the solution of a linear least squares problem in the neighbourhood of A^G.

(ii) Minimization of $F^*(A^*)$. The minimization of the second functional can be performed as follows:

(a) in the neighbourhood of the first guess $(A_1^*, A_2^*, A_3^*) \equiv (x_1^*, x_2^*, x_3^*)$, $N^* > 10$ values of F^* are evaluated,

(b) F^* is then approximated by the polynomial

$$F^* \approx F(a) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_1x_2 + a_5x_1x_3 + a_6x_2x_3 + a_7x_1 + a_8x_2 + a_9x_3 + a_{10}\,. \tag{10}$$

(c) if $a_i > 0$ for $i = 1,..., 6$, the next guess for (x_1^*, x_2^*, x_3^*) is given by the solution of

$$\partial\tilde{F}/\partial x_k = 0; \qquad k = 1, 2, 3; \tag{11}$$

if $a_i \leqslant 0$ for some $i \in (1,..., 6)$, the next guess can be determined using only first derivatives.

The weighting factors w_k in (6) are determined by using the assumption that the first guesses for A and A^* are good for at least small $\Delta\theta$ (i.e., near axis) as follows:

(i) For the grid in Fig. 1b values for $f_k(x_i, \tilde{Y})$ are evaluated with $\Delta\theta^* = \Delta\theta/10$.

(ii) W_k are then determined from the solution of the equations

$$\sum_{i=1,M} w_1^2 f_{1i}^2 = \sum_{i=1,M} w_2^2 f_{2i}^2 = \sum_{i=1,M} w_3^2 f_{3i}^2, \qquad \sum_{k=1,3} w_k^2 = 1. \tag{12}$$

The results obtained using the method described above are compared with those obtained in [1] in Fig. 2.

Conclusions

The following conclusions can be drawn from the above analysis:

(i) Detached shock wave problems can be solved using the least squares approach.

(ii) The calculation time (typically 2 minutes on IBM 360/67 for the case of a sphere in an ideal gas with $M = 24$, $N = 15$ and three iterations for A^*) may be improved by reducing the number of x_i, A_j, A_k^* used, or else using a faster minimization procedure.

(iii) The solution is relatively insensitive to the values of weighting factors w_k, but for $w_1 = w_2 = w_3 = 1$ the procedure is unstable.

(iv) For the case a body of an arbitrary shape, rather than using a complex transformation of independent variables and partial derivatives in (1), the integral form of the conservation laws (e.g., $\iint_S \rho v_n \, dS = 0$) could be used.

In summary, the solution of complex problems in applied physics and engineering can be obtained in a manner similar to that described in this note providing some preliminary knowledge of the form of the solution that is expected is available. In this particular study, the flow in a coaxial plasma accelerator was approximated by quasi-one-dimensional model involving singly and double ionized species [6].

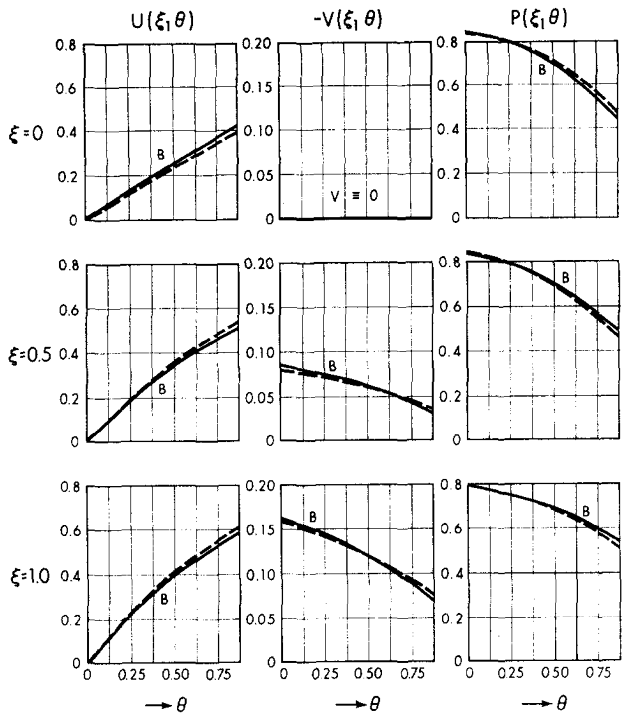

FIG. 2. Comparison of calculated flow profiles for a sphere in a Mach = 10, $\gamma = 1.4$ ideal gas flow. The solid curves are from [1], while the dotted lines represent the present calculation; where $w_1 = 0.15$, $w_2 = 0.61$, $w_3 = 0.77$, three iterations of A^*, $M = 24$, $N = 15$. The resulting shock wave is $r_{\text{shock}} = 1.175 + 0.11\theta^2 + 0.044\theta^4$.

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